On the irrationality of infinite series of reciprocals of square roots

Jaroslav Hančl[1] and Radhakrishnan Nair

November 28, 2015

AMS Class: 11J72.

Key words and phrases: irrationality; infinite series; square roots

[1]This work was supported by the European Regional Development Fund in the IT4Innovations Centre of Excellence project (CZ.1.05/1.1.00/02.0070) and by the grant no. P201/12/2351.

Abstract

This paper gives sufficient conditions on the sequence $\{a_n\}_{n=1}^{\infty}$ to ensure that the number $\sum_{n=1}^{\infty} \frac{1}{\sqrt{a_n}}$ is irrational.

1 Introduction

Following Liouville [12], Mignotte [14] and Erdős [3] we prove the following theorem.

Theorem 1. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a non-decreasing sequence of natural numbers such that*

$$\lim_{n\to\infty} \frac{\log^2 a_n}{2^{n^2}} = \lim_{n\to\infty} a_n^{2^{-n^2/2}} = \infty.$$

Then the number $\sum_{n=1}^{\infty} \frac{1}{\sqrt{a_n}}$ *is irrational.*

Here and through the whole paper $\log x$ means the natural logarithm of the number x. This theorem has some history. In 1975 Erdős [3] proved that if $\{a_n\}_{n=1}^{\infty}$ is a non-decreasing sequence of positive integers such that $\lim_{n\to\infty} a_n^{\frac{1}{2^n}} = \infty$ then the number $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is irrational. Later Hančl [8] proved that if $\{a_n\}_{n=1}^{\infty}$ is a non-decreasing sequence of positive integers such that $1 < \liminf_{n\to\infty} a_n^{\frac{1}{2^n}} \neq \limsup_{n\to\infty} a_n^{\frac{1}{2^n}}$ then the number $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is irrational. After that Šustek [18] found a new irrationality measure for such a number. Also Rucki [16] established a criteria for the sums of reciprocals of natural numbers to be irrational. In 1991 Hančl [6] proved that if $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive real numbers such that $a_n \leq 2^{\frac{1}{n^2}2^n}$ holds for any $n \in \mathbb{N}$ (the natural numbers) then there exists a sequence $\{c_n\}_{n=1}^{\infty}$ of natural numbers such that the number $\sum_{n=1}^{\infty} \frac{1}{c_n a_n}$ is rational.

If $\{a_n\}_{n=1}^{\infty}$ is a sequence of natural numbers such that $a_1 \geq 2$ and $a_{n+1} = a_n^2 - a_n + 1$ for all $n \in \mathbb{N}$ then the number

$$\sum_{n=1}^{\infty} \frac{1}{a_n} = \frac{1}{a_1} + \sum_{n=2}^{\infty} \frac{(a_1-1)\prod_{j=1}^{n-1} a_j}{(a_1-1)\prod_{j=1}^{n} a_j} = \frac{1}{a_1} + \sum_{n=2}^{\infty} \frac{(a_1-1)a_1\prod_{j=2}^{n-1} a_j}{(a_1-1)\prod_{j=1}^{n} a_j} =$$

$$\frac{1}{a_1} + \sum_{n=2}^{\infty} \frac{a_n - 1}{(a_1-1)\prod_{j=1}^{n} a_j} = \frac{1}{a_1} + \sum_{n=2}^{\infty} \left(\frac{1}{(a_1-1)\prod_{j=1}^{n-1} a_j} - \frac{1}{(a_1-1)\prod_{j=1}^{n} a_j}\right) =$$

$$\frac{1}{a_1 - 1}$$

is rational. The sequence $\{a_n^{\frac{1}{2^n}}\}_{n=1}^{\infty}$ is decreasing and all terms are greater than 1. Therefore $\lim_{n\to\infty} a_n^{\frac{1}{2^n}}$ exists. Aho and Sloane [1] proved that if $a_0 = 2$ then $a_n \doteq 1.264^{2^n}$. See also Finch [4], page 444. In the following we give some bounds for $\lim_{n\to\infty} a_n^{\frac{1}{2^n}}$. We have $a_2 = a_1^2 - a_1 + 1$, $a_3 = (a_1^2 - a_1 + 1)a_1(a_1 - 1) + 1$. By induction we can prove that $(a_1^2 - a_1 + 1)^{2^{n-2}} - (a_1^2 - a_1 + 1)^{2^{n-3}} + 1 \geq a_n \geq (a_1^2 - a_1)^{2^{n-2}} + 1$ for every positive integer $n \geq 3$. Hence $\sqrt[4]{a_1^2 - a_1 + 1} \geq \lim_{n\to\infty} a_n^{\frac{1}{2^n}} \geq \sqrt[4]{a_1^2 - a_1} > 1$. This implies that the condition $\lim_{n\to\infty} a_n^{\frac{1}{2^n}} = \infty$ or even something weaker with additional assumptions is necessary for the irrationality of $\sum_{n=1}^{\infty} \frac{1}{a_n}$.

Denote in the whole paper $\mathbb{N}$ and $\mathbb{Z}$ the set of all positive integers and integers, respectively. Recall that the number α is a Liouville number if for every $n \in \mathbb{N}$ the inequality $\mid \alpha - \frac{p}{q} \mid < \frac{1}{q^n}$ has infinitely many solutions in $(p, q) \in \mathbb{Z} \times \mathbb{N}$. Erdős [3] proved that if $\{a_n\}_{n=1}^{\infty}$ is a sequence of natural numbers such that $\lim_{n\to\infty} \frac{1}{n} \log\log a_n = \infty$ then the number $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is a Liouville number. Some other conditions for series to be Liouville numbers can be found in [5]. Kanoko, Kurosawa and Shiokawa [11] proved the transcendence of reciprocal sums of elements in some binary recurrence sequences. On the other side Lucas [13] proved that $\sum_{n=1}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2}$ where $\{F_n\}_{n=1}^{\infty}$ is the increasing sequence of all Fibonacci numbers. Hančl [7] proved that if $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive integers such that $\lim_{n\to\infty} \frac{1}{n} \log_3 \log_2 a_n > 1$ then the number $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is transcendental. Here and through the whole paper $\log_a x$ means the logarithm on the base a of the number x. The authors are not able to find a sequence $\{a_n\}_{n=1}^{\infty}$ of natural numbers such that $\lim_{n\to\infty} \frac{1}{n} \log_2 \log_2 a_n > 1$ and that the number $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is algebraic.

The main result of this paper is Theorem 5 which gives conditions on the sequence $\{a_n\}_{n=1}^{\infty}$ under which the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{a_n}}$ is an irrational number.

Its proof is based on an idea of Erdős [3] and Liouville [12]. The conditions on the sequence $\{a_n\}_{n=1}^{\infty}$ are quite general. In particular it is not required that the elements of $\{a_n\}_{n=1}^{\infty}$ be approximable by the elements of a finite union of power sequences or be associated with any differential equation. This means we cannot rely on the Main Theorem from the paper of Corvaja and Zannier [2] which uses the Subspace method or Theorem 1, page 34 from Nishioka' s book [15] dealing with the method of Mahler.

2 Notation and preliminary results

Let α be an algebraic number with minimal polynomial $P(x) = \sum_{j=0}^{d} a_j x^j$ and conjugates $\alpha = \alpha_1, \ldots, \alpha_d$. Then the Mahler measure $M(\alpha)$ of the number α is defined $M(\alpha) :=\mid a_d \mid \prod_{j=1}^{d} \max(1, \mid \alpha_j \mid)$. Set $H(\alpha) = M(\alpha)^{\frac{1}{d}}$. Now we have the following lemma.

Lemma 2. *Let n be a positive integer and let $\beta_1, \ldots, \beta_n$ be algebraic numbers. Then*

$$H(\sum_{j=1}^{n} \beta_j) \leq 2^n \prod_{j=1}^{n} H(\beta_j) \tag{1}$$

and

$$\deg(\sum_{j=1}^{n} \beta_j) \leq 2^n \prod_{j=1}^{n} \deg(\beta_j). \tag{2}$$

For the proof of (1) see Waldschmidt [19], Property 3.3, page 75 together with Lemma 3.10, page 79. See also Stewart [17]. Proof of (2) can be found in Isaacs [10].

We also need the following theorem [14] and lemma [9].

Theorem 3. *Let α and β be different algebraic numbers of degree A and B, respectively. Then*

$$\mid \alpha - \beta \mid \geq \frac{1}{2^{AB} M(\alpha)^B M(\beta)^A}. \tag{3}$$

Lemma 4. *Suppose $\varepsilon > 0$ and let $\{b_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive real numbers such that $b_n \geq n^{1+\varepsilon}$ for all $n \in \mathbb{N}$. Then for every $N \geq 1$ we have*

$$\sum_{n=N}^{\infty} \frac{1}{b_n} < \frac{1 + \frac{2^{\varepsilon}}{\varepsilon}}{b_N^{\frac{\varepsilon}{1+\varepsilon}}}. \tag{4}$$

Our main result is the following theorem.

Theorem 5. *Suppose $\varepsilon > 0$ and let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive integers such that*

$$\limsup_{n\to\infty} a_n^{\frac{1}{\prod_{j=1}^{n-1}(2^j+2)}} = \infty \tag{5}$$

and such that

$$a_n \geq n^{2+\varepsilon} \tag{6}$$

for all sufficiently large n. Then $\sum_{n=1}^{\infty} \frac{1}{\sqrt{a_n}}$ is irrational.

3 Proofs

Theorem 1 is an immediate consequence of Theorem 5. We now prove Theorem 5.

Proof. Suppose that there exist $p, q \in \mathbb{N}$ such that $\gamma = \sum_{n=1}^{\infty} \frac{1}{\sqrt{a_n}} = \frac{p}{q}$. Set $\gamma_N = \sum_{n=1}^{N} \frac{1}{\sqrt{a_n}}$. Then we have $M(\gamma) = \max(p, q)$, $\deg(\gamma_N) \leq 2^N$ and

$$M(\gamma_N) = H(\gamma_N)^{\deg(\gamma_N)} \leq H(\gamma_N)^{2^N} \leq (2^N \prod_{n=1}^{N} H(\frac{1}{\sqrt{a_n}}))^{2^N} \leq (2^N \prod_{n=1}^{N} \sqrt{a_n})^{2^N}.$$

From this and Theorem 3 we obtain that

$$\gamma(N) = \mid \gamma - \gamma_N \mid \geq \frac{1}{2^{\deg(\gamma)\deg(\gamma_N)} M(\gamma)^{\deg(\gamma_N)} M(\gamma_N)^{\deg(\gamma)}} \geq$$

$$\frac{1}{(2\max(p,q))^{2^N} M(\gamma_N)} \geq \frac{1}{(\max(p,q)2^{N+1} \prod_{n=1}^{N} \sqrt{a_n})^{2^N}}.$$

Hence for all sufficiently large N we have

$$\gamma(N)(\max(p,q)2^{N+1}\prod_{n=1}^{N}\sqrt{a_n})^{2^N} \geq 1. \tag{7}$$

Now the proof falls into three cases.

1. Assume that

$$\limsup_{n\to\infty} a_n^{\frac{1}{\prod_{j=1}^{n-1}(3^j+3)}} = \infty. \tag{8}$$

It then follows for infinitely many N, that we have

$$a_{N+1}^{\frac{1}{\prod_{j=1}^{N}(3^j+3)}} \geq (1+\frac{1}{(N+1)^2})\max_{n=1,\dots N} a_n^{\frac{1}{\prod_{j=1}^{n-1}(3^j+3)}} \tag{9}$$

for otherwise there would exist N_0 such that for every $N \geq N_0$ we have

$$a_{N+1}^{\frac{1}{\prod_{j=1}^{N}(3^j+3)}} < (1+\frac{1}{(N+1)^2})\max_{n=1,\dots N} a_n^{\frac{1}{\prod_{j=1}^{n-1}(3^j+3)}} <$$

$$(1+\frac{1}{(N+1)^2})(1+\frac{1}{N^2})\max_{n=1,\dots N-1} a_n^{\frac{1}{\prod_{j=1}^{n-1}(3^j+3)}} < \cdots <$$

$$\prod_{n=N_0+1}^{N+1}(1+\frac{1}{n^2})\max_{n=1,\dots N_0} a_n^{\frac{1}{\prod_{j=1}^{n-1}(3^j+3)}} <$$

$$\prod_{n=1}^{\infty}(1+\frac{1}{n^2})\max_{n=1,\dots N_0} a_n^{\frac{1}{\prod_{j=1}^{n-1}(3^j+3)}} = const$$

which contradicts (8). From (9) we obtain that for infinitely many N,

$$a_{N+1} \geq ((1+\frac{1}{(N+1)^2})\max_{n=1,\dots N} a_n^{\frac{1}{\prod_{j=1}^{n-1}(3^j+3)}})^{\prod_{j=1}^{N}(3^j+3)} =$$

$$(1+\frac{1}{(N+1)^2})^{\prod_{j=1}^{N}(3^j+3)}(\max_{n=1,\dots N} a_n^{\frac{1}{\prod_{j=1}^{n-1}(3^j+3)}})^{\prod_{j=1}^{N}(3^j+3)} >$$

$$2^{3^N}(\max_{n=1,\dots N} a_n^{\frac{1}{\prod_{j=1}^{n-1}(3^j+3)}})^{\prod_{j=1}^{N}(3^j+3)} =$$

$$(2(\max_{n=1,\dots N} a_n^{\frac{1}{\prod_{j=1}^{n-1}(3^j+3)}})^{\prod_{j=1}^{N-1}(3^j+3)+3^{-(N-1)}\prod_{j=1}^{N-1}(3^j+3)})^{3^N} \geq$$

$$(2a_N(\max_{n=1,\dots N} a_n^{\frac{1}{\prod_{j=1}^{n-1}(3^j+3)}})^{3^{-(N-1)}\prod_{j=1}^{N-1}(3^j+3)})^{3^N} \geq \cdots \geq (2\prod_{j=1}^{N} a_j)^{3^N}.$$

This and Lemma 4 yield

$$\gamma(N)(\max(p,q)2^{N+1}\prod_{n=1}^{N}\sqrt{a_n})^{2^N} \leq \frac{1+\frac{2^{\frac{\varepsilon}{2}+1}}{\varepsilon}}{a_{N+1}^{\frac{\varepsilon}{4+2\varepsilon}}}(\max(p,q)2^{N+1}\prod_{n=1}^{N}\sqrt{a_n})^{2^N} \leq$$

$$\frac{1+\frac{2^{\frac{\varepsilon}{2}+1}}{\varepsilon}}{((2\prod_{j=1}^{N} a_j)^{3^N})^{\frac{\varepsilon}{4+2\varepsilon}}}(\max(p,q)2^{N+1}\prod_{n=1}^{N}\sqrt{a_n})^{2^N} < 1$$

for infinitely many N. This contradicts (7).

2. Suppose that

$$\limsup_{n\to\infty} a_n^{\frac{1}{\prod_{j=1}^{n-1}(3^j+3)}} < \infty \tag{10}$$

and for all large n,

$$a_n \geq 2^n. \tag{11}$$

From (10) we obtain that for every large n,

$$a_n < 2^{3^{n^2}}. \tag{12}$$

Inequality (11) yields that for every large N,

$$\sum_{n=N+1}^{\infty}\frac{1}{\sqrt{a_n}} = \sum_{n\leq \log_2 a_{N+1}}\frac{1}{\sqrt{a_n}} + \sum_{n>\log_2 a_{N+1}}\frac{1}{\sqrt{a_n}} \leq$$

$$\frac{\log_2 a_{N+1}}{\sqrt{a_{N+1}}} + \sum_{n>\log_2 a_{N+1}}\frac{1}{\sqrt{2^n}} \leq \frac{\log_2 a_{N+1}}{\sqrt{a_{N+1}}} + \sum_{n=0}^{\infty}\frac{1}{\sqrt{a_{N+1}}\sqrt{2^n}} < \frac{2\log_2 a_{N+1}}{\sqrt{a_{N+1}}}.$$

This and (12) imply that for every large N we have

$$\sum_{n=N+1}^{\infty}\frac{1}{\sqrt{a_n}} < \frac{4^{N^2}}{\sqrt{a_{N+1}}}. \tag{13}$$

Now from (5) we obtain that for infinitely many N we have

$$a_{N+1}^{\frac{1}{\prod_{j=1}^{N}(2^j+2)}} \geq (1+\frac{1}{(N+1)^2})\max_{n=1,\dots N} a_n^{\frac{1}{\prod_{j=1}^{n-1}(2^j+2)}}, \tag{14}$$

otherwise, as before, there would exist N_0 such that for every $N \geq N_0$ we have

$$a_{N+1}^{\frac{1}{\prod_{j=1}^{N}(2^j+2)}} < (1 + \frac{1}{(N+1)^2}) \max_{n=1,\dots N} a_n^{\frac{1}{\prod_{j=1}^{n-1}(2^j+2)}} <$$

$$(1 + \frac{1}{(N+1)^2})(1 + \frac{1}{N^2}) \max_{n=1,\dots N-1} a_n^{\frac{1}{\prod_{j=1}^{n-1}(2^j+2)}} < \dots <$$

$$\prod_{n=N_0+1}^{N+1} (1 + \frac{1}{n^2}) \max_{n=1,\dots N_0} a_n^{\frac{1}{\prod_{j=1}^{n-1}(2^j+2)}} <$$

$$\prod_{n=1}^{\infty}(1 + \frac{1}{n^2}) \max_{n=1,\dots N_0} a_n^{\frac{1}{\prod_{j=1}^{n-1}(2^j+2)}} = const.$$

This contradicts (3). From (14) we obtain that for infinitely many N we have

$$a_{N+1} \geq ((1 + \frac{1}{(N+1)^2}) \max_{n=1,\dots N} a_n^{\frac{1}{\prod_{j=1}^{n-1}(2^j+2)}})^{\prod_{j=1}^{N}(2^j+2)} =$$

$$(1 + \frac{1}{(N+1)^2})^{\prod_{j=1}^{N}(2^j+2)}(\max_{n=1,\dots N} a_n^{\frac{1}{\prod_{j=1}^{n-1}(3^j+3)}})^{\prod_{j=1}^{N}(2^j+2)} >$$

$$2^{N^2 2^N}(\max_{n=1,\dots N} a_n^{\frac{1}{\prod_{j=1}^{n-1}(2^j+2)}})^{\prod_{j=1}^{N}(2^j+2)} =$$

$$(2^{N^2}(\max_{n=1,\dots N} a_n^{\frac{1}{\prod_{j=1}^{n-1}(3^j+3)}})^{\prod_{j=1}^{N-1}(2^j+2)+3^{-(N-1)}\prod_{j=1}^{N-1}(3^j+3)})^{2^N} \geq$$

$$(2^{N^2} a_N(\max_{n=1,\dots N} a_n^{\frac{1}{\prod_{j=1}^{n-1}(3^j+3)}})^{3^{-(N-1)}\prod_{j=1}^{N-1}(2^j+2)})^{2^N} \geq \dots \geq (2^{N^2}\prod_{j=1}^{N} a_j)^{2^N}.$$

This and (13) imply for infinitely many N that we have

$$\gamma(N)(\max(p,q)2^{N+1}\prod_{n=1}^{N}\sqrt{a_n})^{2^N} =$$

$$(\sum_{n=N+1}^{\infty}\frac{1}{\sqrt{a_n}})(\max(p,q)2^{N+1}\prod_{n=1}^{N}\sqrt{a_n})^{2^N} \leq$$

$$(\frac{4^{N^2}}{\sqrt{a_{N+1}}})(\max(p,q)2^{N+1}\prod_{n=1}^{N}\sqrt{a_n})^{2^N} \leq$$

$$(\frac{4^{N^2}}{\sqrt{(2^{N^2}\prod_{j=1}^{N}a_j)^{2^N}}})(\max(p,q)2^{N+1}\prod_{n=1}^{N}\sqrt{a_n})^{2^N}<1$$

which contradicts (7).

3. Suppose that (10) holds. Suppose in addition that for infinitely many n the inequality

$$a_n \leq 2^n. \tag{15}$$

also holds. Then (12) holds for every large n. Assume that B is a sufficiently large positive real number. From (5) we obtain that there exists a least integer S such that

$$a_S \geq 2^{B\prod_{j=1}^{S-1}(2^j+2)}. \tag{16}$$

Let K be the greatest integer less than S such that (15) holds. Let R be the least integer greater than K such that

$$a_R > ((1+\frac{1}{R^2})\max_{n=K,\ldots R-1} a_n^{\frac{1}{\prod_{j=1}^{n-1}(2^j+2)}})^{\prod_{j=1}^{R-1}(2^j+2)} \tag{17}$$

and that

$$a_s \leq ((1+\frac{1}{s^2})\max_{n=K,\ldots s-1} a_n^{\frac{1}{\prod_{j=1}^{n-1}(2^j+2)}})^{\prod_{j=1}^{s-1}(2^j+2)} \tag{18}$$

for all $s = K+1,\ldots,R-1$. Note that $R \leq S$ because otherwise (15), (16) and (18) would imply that

$$2^B \leq a_S^{\frac{1}{\prod_{j=1}^{S-1}(2^j+2)}} \leq (1+\frac{1}{S^2})\max_{n=K,\ldots S-1} a_n^{\frac{1}{\prod_{j=1}^{n-1}(2^j+2)}} \leq \cdots <$$

$$(\prod_{n=1}^{\infty}(1+\frac{1}{n^2}))a_K^{\frac{1}{\prod_{j=1}^{K-1}(2^j+2)}} < 2(\prod_{n=1}^{\infty}(1+\frac{1}{n^2})) = const.$$

This is a contradiction for large B. From (15), (17) and the fact that $\{a_n\}_{n=1}^{\infty}$ is a non-decreasing sequence we obtain that

$$a_R > ((1+\frac{1}{R^2})\max_{n=K,\ldots R-1} a_n^{\frac{1}{\prod_{j=1}^{n-1}(2^j+2)}})^{\prod_{j=1}^{R-1}(2^j+2)} =$$

$$(1+\frac{1}{R^2})^{\prod_{j=1}^{R-1}(2^j+2)}(\max_{n=K,\dots R-1} a_n^{\frac{1}{\prod_{j=1}^{n-1}(2^j+2)}})^{\prod_{j=1}^{R-1}(2^j+2)} \geq$$

$$(1+\frac{1}{R^2})^{\prod_{j=1}^{R-1}(2^j+2)}(a_{R-1}(\max_{n=K,\dots R-1} a_n^{\frac{1}{\prod_{j=1}^{n-1}(2^j+2)}})^{2^{-(R-2)}\prod_{j=1}^{R-2}(2^j+2)})^{2^{R-1}} \geq \dots \geq$$

$$(1+\frac{1}{R^2})^{\prod_{j=1}^{R-1}(2^j+2)}(\prod_{j=K+1}^{R-1} a_j)^{2^{R-1}} \geq 2^{2^{4R}}(\prod_{j=1}^{R-1} a_j)^{2^{R-1}}. \tag{19}$$

Now inequality (18) yields, for all $s = K+1, \dots, R-1$, that we have

$$a_s^{\frac{1}{\prod_{j=1}^{s-1}(2^j+2)}} \leq (1+\frac{1}{s^2})\max_{n=K,\dots s-1} a_n^{\frac{1}{\prod_{j=1}^{n-1}(2^j+2)}} \leq$$

$$(1+\frac{1}{s^2})(1+\frac{1}{(s-1)^2})\max_{n=K,\dots s-2} a_n^{\frac{1}{\prod_{j=1}^{n-1}(2^j+2)}} \leq \dots \leq$$

$$(\prod_{j=1}^{\infty}(1+\frac{1}{j^2}))a_K^{\frac{1}{\prod_{j=1}^{K-1}(2^j+2)}} \leq D$$

where D is a constant which does not depend on K. Hence

$$\prod_{s=1}^{R-1} a_s = (\prod_{s=1}^{K} a_s)(\prod_{s=K+1}^{R-1} a_s) \leq 2^{K^2}\prod_{s=K+1}^{R-1} D^{\prod_{j=1}^{s-1}(2^j+2)} < D^{2\prod_{j=1}^{R-2}(2^j+2)}. \tag{20}$$

From Lemma 4, (12), and the fact that $a_n \geq 2^n$ for every $n = K+1, \dots, S$ we obtain

$$\sum_{n=R}^{\infty}\frac{1}{\sqrt{a_n}} = \sum_{n \leq \log_2 a_R}\frac{1}{\sqrt{a_n}} + \sum_{S>n>\log_2 a_R}\frac{1}{\sqrt{a_n}} + \sum_{n=S}^{\infty}\frac{1}{\sqrt{a_n}} \leq$$

$$\frac{\log_2 a_R}{\sqrt{a_R}} + \sum_{n>\log_2 a_R}\frac{1}{\sqrt{2^n}} + \frac{1+\frac{2^{\frac{\varepsilon}{2}+1}}{\varepsilon}}{a_S^{\frac{\varepsilon}{4+2\varepsilon}}} \leq$$

$$\frac{\log_2 a_R}{\sqrt{a_R}} + \sum_{n=1}^{\infty}\frac{1}{\sqrt{a_R}\sqrt{2^n}} + \frac{1}{a_S^{\frac{\varepsilon}{4+4\varepsilon}}} < \frac{2\log_2 a_R}{\sqrt{a_R}} + \frac{1}{a_S^{\frac{\varepsilon}{4+4\varepsilon}}}.$$

This, (12), (16) and (19) imply

$$\sum_{n=R}^{\infty}\frac{1}{\sqrt{a_n}} < \frac{2\log_2 a_R}{\sqrt{a_R}} + \frac{1}{a_S^{\frac{\varepsilon}{4+4\varepsilon}}} < \frac{3^{R^3}}{\sqrt{2^{2^{4R}}(\prod_{j=1}^{R-1} a_j)^{2^{R-1}}}} + \frac{1}{2^{\frac{\varepsilon}{4+4\varepsilon}B\prod_{j=1}^{S-1}(2^j+2)}} <$$

$$\frac{1}{2^{2^{3R}}(\prod_{j=1}^{R-1} a_j)^{2^{R-2}}} + \frac{1}{2^{\frac{\varepsilon}{4+4\varepsilon} B \prod_{j=1}^{S-1}(2^j+2)}}.$$

From this and (20) we obtain that for a sufficiently large B,

$$\gamma(R-1)(\max(p,q)2^R \prod_{n=1}^{R-1} \sqrt{a_n})^{2^{R-1}} = (\sum_{n=R}^{\infty} \frac{1}{\sqrt{a_n}})(\max(p,q)2^R \prod_{n=1}^{R-1} \sqrt{a_n})^{2^{R-1}} \leq$$

$$(\frac{1}{2^{2^{3R}}(\prod_{j=1}^{R-1} a_j)^{2^{R-2}}} + \frac{1}{2^{\frac{\varepsilon}{4+4\varepsilon} B \prod_{j=1}^{S-1}(2^j+2)}})(\max(p,q)2^R \prod_{n=1}^{R-1} \sqrt{a_n})^{2^{R-1}} =$$

$$\frac{(\max(p,q)2^R \prod_{n=1}^{R-1} \sqrt{a_n})^{2^{R-1}}}{2^{2^{3R}}(\prod_{j=1}^{R-1} a_j)^{2^{R-2}}} + \frac{(\max(p,q)2^R \prod_{n=1}^{R-1} \sqrt{a_n})^{2^{R-1}}}{2^{\frac{\varepsilon}{4+4\varepsilon} B \prod_{j=1}^{S-1}(2^j+2)}} \leq$$

$$\frac{(\max(p,q)2^R \prod_{n=1}^{R-1} \sqrt{a_n})^{2^{R-1}}}{2^{2^{3R}}(\prod_{j=1}^{R-1} a_j)^{2^{R-2}}} + \frac{(\max(p,q)2^R D^{\prod_{j=1}^{R-2}(2^j+2)})^{2^{R-1}}}{2^{\frac{\varepsilon}{4+4\varepsilon} B \prod_{j=1}^{S-1}(2^j+2)}} < 1.$$

This contradicts (7).

$\square$

References

[1] Aho A. V. and Sloane N. J. A.: Some doubly exponential sequences, Fibonacci Quart. 11, no. 4, (1973), 429–437.

[2] Corvaja P., Zannier U.: On the rational approximation to the powers of an algebraic number: Solution of two problems of Mahler and Mendès France, Acta Math. 193, (2004), 175–191.

[3] Erdős P.: Some problems and results on the irrationality of the sum of infinite series, J. Math. Sci. 10, (1975), 1–7.

[4] Finch S.: Mathematical constants, Encyclopedia of Mathematics and its Applications, 94, Cambridge University Press, Cambridge, (2003).

[5] Gözeri K., Pekin A., Kilicman A.,: On the transcendence of some power seriws, Adv. Difference Equ., (2013), 2013: 17, 8 pp.

[6] Hančl J. : Expression of Real Numbers with Help of Infinite Series, Acta Arithmetica, LIX.2 (1991), 97–104.

[7] Hančl J. : Two Criteria for Transcendental Sequences, Le Matematiche, vol. LVI, no. 1, (2001), 149–160.

[8] Hančl J. : A criterion for linear independence of series, Rocky Mountain J. Math., vol. 34, no. 1, (2004), 173–186.

[9] Hančl J., Pulcerová S., Kolouch O., Štěpnička J. : A note to the transcendence of infinite products, Czech. Math. J., Vol. 62, no. 3, (2012), 613–623.

[10] Isaacs I. M.: Degree of sums in a separable field extension, Proc. Amer. Math. Soc. 25, (1970), 638–641.

[11] Kanoko T., Kurosawa T., Shiokawa I.: Trancendence of reciprocal sums of binary recurrences, Monatsch. Math. 157, no. 4, (2009), 323–334.

[12] Liouville J.: Nouvelle démonstration d' un théoréme sur les irrationelles algébriques, C. R. Acad. Sci. Paris 18, (1844), 910–911.

[13] Lucas E.: Théorie des fonctions numériques simplement périodiques, Am. J. Math. 1, (1878), 184–240.

[14] Mignotte M.: Approximation des nombres algébriques par des nombres algébriques de grand degré, (French), Ann. Fac. Sci. Toulouse Math. (5) 1, no. 2, (1979), 165–170.

[15] Nishioka K.: Mahler Functions and Transcendence, Lecture notes in mathematics 1631, Springer, (1996).

[16] Rucki P.: Irrationality of infinite series of a special kind, Mathematica Slovaca, (to appear).

[17] Stewart C. L.: On heigts of multiplicatively dependent algebraic numbers, Acta Arith. 133, no. 2, (2008), 97–108.

[18] Šustek J.: New bounds for irrationality measures of some fast converging series, Mathematics for Applications, vol. 3, no. 2, (2014).

[19] Waldschmidt M.: Diophantine approximation on linear algebraic groups, Transcendence properties of the exponential function several variables, Grundlehren der mathematischen wissenschaften, Fundamental principles of mathematical sciences 326, Springler Verlag, Berlin, (2000).

J. Hančl,

Department of Mathematics and Centre of Excellence IT4Innovation, division of UO, Institute for Research and Applications of Fuzzy Modeling, University of Ostrava, 30. dubna 22, 701 03 Ostrava 1, Czech Republic.

e-mail: hancl@osu.cz,

R. Nair,

Mathematical Sciences,

The University of Liverpool,

Peach Street, Liverpool L69 7ZL, United Kingdom,

e-mail: nair@liverpool.ac.uk.